

Fractal statistics, fractal index and fractons

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Abstract

The concept of fractal index is introduced in connection with the idea of universal class h of particles or quasiparticles, termed fractons, which obey fractal statistics. We show the relation between fractons and conformal field theory (CFT)-quasiparticles taking into account the central charge $c[\nu]$ and the particle-hole duality $\nu \longleftrightarrow \frac{1}{\nu}$, for integer-value ν of the statistical parameter. The Hausdorff dimension h which labelled the universal classes of particles and the conformal anomaly are therefore related. We also establish a connection between Rogers dilogarithm function, Farey series of rational numbers and the Hausdorff dimension.

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We consider the conformal field theory(CFT)-quasiparticles (edge excitations) in connection with the concept of fractons introduced in [1]. These excitations have been considered at the edge of the quantum Hall systems which in the fractional regime assume the form of a chiral Luttinger liquid [2]. Beyond this, conformal field theories have been exploited in a variety of contexts, including statistical mechanics at the critical point, field theories, string theory, and in various branches of mathematics [3].

In this Letter, we suppose that the fractal statistics obeyed by fractons are shared by CFT-quasiparticles. Thus, the central charge, a model dependent constant is related to the universal class h of the fractons. We define the *fractal index* associated with these classes as

$$i_f[h] = \frac{6}{\pi^2} \int_{\infty(T=0)}^{1(T=\infty)} \frac{d\xi}{\xi} \ln \{\Theta[\mathcal{Y}(\xi)]\} \quad (1)$$

where

$$\Theta[\mathcal{Y}] = \frac{\mathcal{Y}[\xi] - 2}{\mathcal{Y}[\xi] - 1} \quad (2)$$

is the single-particle partition function of the universal class h and $\xi = \exp \{(\epsilon - \mu)/KT\}$, has the usual definition. The function $\mathcal{Y}[\xi]$ satisfies the equation

$$\xi = \{\mathcal{Y}[\xi] - 1\}^{h-1} \{\mathcal{Y}[\xi] - 2\}^{2-h}. \quad (3)$$

We note that the general solution of the algebraic equation derived from this last one is of the form

$$\mathcal{Y}_h[\xi] = f[\xi] + \tilde{h}$$

or

$$\mathcal{Y}_{\tilde{h}}[\xi] = g[\xi] + h,$$

where $\tilde{h} = 3 - h$, is a *duality symmetry*¹ between the classes. The functions $f[\xi]$ and $g[\xi]$ at least for third, fourth degrees algebraic equation differ by plus and minus signs in some terms of their expressions.

The particles within each class h satisfy specific **fractal statistics**² [1]

$$\begin{aligned} n &= \xi \frac{\partial}{\partial \xi} \ln \Theta[\mathcal{Y}] \\ &= \frac{1}{\mathcal{Y}[\xi] - h} \end{aligned} \quad (4)$$

¹This means that fermions($h = 1$) and bosons($h = 2$) are dual objects. As a result we have a *fractal supersymmetry*, since for the particle with spin s within the class h , its dual $s + \frac{1}{2}$ is within the class \tilde{h} .

²In fact, we have here *fractal functions* [4].

and the fractal parameter ³ (or Hausdorff dimension) h defined in the interval $1 < h < 2$ is related to the spin-statistics relation $\nu = 2s$ through the *fractal spectrum*

$$h - 1 = 1 - \nu, \quad 0 < \nu < 1; \quad h - 1 = \nu - 1, \quad 1 < \nu < 2; \quad (5)$$

etc.

For $h = 1$ we have fermions, with $\mathcal{Y}[\xi] = \xi + 2$, $\Theta[1] = \frac{\xi}{\xi+1}$ and $i_f[1] = \frac{6}{\pi^2} \int_{\infty}^1 \frac{d\xi}{\xi} \ln \left\{ \frac{\xi}{\xi+1} \right\} = \frac{1}{2}$. For $h = 2$ we have bosons, with $\mathcal{Y}[\xi] = \xi + 1$, $\Theta[2] = \frac{\xi-1}{\xi}$ and $i_f[2] = \frac{6}{\pi^2} \int_{\infty}^1 \frac{d\xi}{\xi} \ln \left\{ \frac{\xi-1}{\xi} \right\} = 1$. On the other hand, for the universal class $h = \frac{3}{2}$, we have fractons with $\mathcal{Y}[\xi] = \frac{3}{2} + \sqrt{\frac{1}{4} + \xi^2}$, $\Theta \left[\frac{3}{2} \right] = \frac{\sqrt{1+4\xi^2}-1}{\sqrt{1+4\xi^2}+1}$ and $i_f \left[\frac{3}{2} \right] = \frac{6}{\pi^2} \int_{\infty}^1 \frac{d\xi}{\xi} \ln \left\{ \frac{\sqrt{1+4\xi^2}-1}{\sqrt{1+4\xi^2}+1} \right\} = \frac{3}{5}$.

The distribution function for each class h above are given by

$$n[1] = \frac{1}{\xi + 1}, \quad (6)$$

$$n[2] = \frac{1}{\xi - 1}, \quad (7)$$

$$n \left[\frac{3}{2} \right] = \frac{1}{\sqrt{\frac{1}{4} + \xi^2}}, \quad (8)$$

i.e. we have the Fermi-Dirac distribution, the Bose-Einstein distribution and the fracton distribution of the universal class $h = \frac{3}{2}$, respectively. Thus, our formulation generalizes *in a natural way* the fermionic and bosonic distributions for particles assuming rational or irrational values for the spin quantum number s . In this way, our approach can be understood as a *quantum-geometrical* description of the statistical laws of Nature. This means that the (Eq.4) captures the observation about the fractal characteristic of the *quantum-mechanical* path, which reflects the Heisenberg uncertainty principle.

The fractal index as defined has a connection with the central charge or conformal anomaly $c[\nu]$, a dimensionless number which characterizes conformal field theories in two dimensions. This way, we verify that the conformal anomaly is associated with universality classes, i.e. universal classes h of particles. Now, we consider the particle-hole duality $\nu \longleftrightarrow \frac{1}{\nu}$ for integer-value ν of the statistical parameter in connection with the universal class h . For bosons and fermions, we have

$$\{0, 2, 4, 6, \dots\}_{h=2}$$

and

$$\{1, 3, 5, 7, \dots\}_{h=1}$$

such that, the central charge for ν *even* is defined by

³This parameter describes the properties of the path (*fractal curve*) of the quantum-mechanical particle.

$$c[\nu] = i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right] \quad (9)$$

and for ν odd is defined by

$$c[\nu] = 2 \times i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right], \quad (10)$$

where $i_f[h, \nu]$ means the fractal index of the universal class h which contains the particles with distinct spin values which obey specific fractal statistics. We assume that the fractal index $i_f[h, \infty] = 0$ and we obtain, for example, the results

$$\begin{aligned} c[0] &= i_f[2, 0] - i_f[h, \infty] = 1; \\ c[1] &= 2 \times i_f[1, 1] - i_f[1, 1] = \frac{1}{2}; \\ c[2] &= i_f[2, 2] - i_f\left[\frac{3}{2}, \frac{1}{2}\right] = 1 - \frac{3}{5} = \frac{2}{5}; \\ c[3] &= 2 \times i_f[1, 3] - i_f\left[\frac{5}{3}, \frac{1}{3}\right] = 1 - 0.656 = 0.344; \\ &etc, \end{aligned} \quad (11)$$

where the fractal index for $h = \frac{5}{3}$ is obtained from

$$\begin{aligned} i_f\left[\frac{5}{3}\right] &= \frac{6}{\pi^2} \int_{\infty}^1 \frac{d\xi}{\xi} \\ &\times \ln \left\{ \frac{\sqrt[3]{\frac{1}{27} + \frac{\xi^3}{2} + \frac{1}{18}\sqrt{12\xi^3 + 81\xi^6}} + \frac{1}{9\sqrt[3]{\frac{1}{27} + \frac{\xi^3}{2} + \frac{1}{18}\sqrt{12\xi^3 + 81\xi^6}}} - \frac{2}{3}}{\sqrt[3]{\frac{1}{27} + \frac{\xi^3}{2} + \frac{1}{18}\sqrt{12\xi^3 + 81\xi^6}} + \frac{1}{9\sqrt[3]{\frac{1}{27} + \frac{\xi^3}{2} + \frac{1}{18}\sqrt{12\xi^3 + 81\xi^6}}} + \frac{1}{3}} \right\} \\ &= 0.656 \end{aligned} \quad (12)$$

and for its dual we have

$$\begin{aligned} i_f\left[\frac{4}{3}\right] &= \frac{6}{\pi^2} \int_{\infty}^1 \frac{d\xi}{\xi} \\ &\times \ln \left\{ \frac{\sqrt[3]{-\frac{1}{27} + \frac{\xi^3}{2} + \frac{1}{18}\sqrt{-12\xi^3 + 81\xi^6}} + \frac{1}{9\sqrt[3]{-\frac{1}{27} + \frac{\xi^3}{2} + \frac{1}{18}\sqrt{-12\xi^3 + 81\xi^6}}} - \frac{1}{3}}{\sqrt[3]{-\frac{1}{27} + \frac{\xi^3}{2} + \frac{1}{18}\sqrt{-12\xi^3 + 81\xi^6}} + \frac{1}{9\sqrt[3]{-\frac{1}{27} + \frac{\xi^3}{2} + \frac{1}{18}\sqrt{-12\xi^3 + 81\xi^6}}} + \frac{2}{3}} \right\} \\ &= 0.56. \end{aligned} \quad (13)$$

The correlation between the classes h of particles and their fractal index, show us a *robust* consistence in accordance with the unitary $c[\nu] < 1$ representations [3]. Therefore, since h is defined within the interval $1 < h < 2$, the corresponding fractal index is into the interval $0.5 < i_f[h] < 1$. However, the central charge $c[\nu]$ can assumes values less than 0.5. Thus, we distinguish two concepts of central charge, one is related to the universal classes h and the other is related to the particles which belong to these classes.

For the statistical parameter in the interval $0 < \nu < 1$ (the first elements of each class h), $c[\nu] = i_f[h, \nu]$, as otherwise we obtain different values. In another way, the central charge $c[\nu]$ can be obtained using the Rogers dilogarithm function [6], i.e.

$$c[\nu] = \frac{L[x^\nu]}{L[1]}, \quad (14)$$

with $x^\nu = 1 - x$, $\nu = 0, 1, 2, 3, \text{etc.}$ and

$$L[x] = -\frac{1}{2} \int_0^x \left\{ \frac{\ln(1-y)}{y} + \frac{\ln y}{1-y} \right\} dy, \quad 0 < x < 1. \quad (15)$$

This way, we observe that our formulation to the universal class h of particles with any values of spin s establishes a connection between Hausdorff dimension h and the central charge $c[\nu]$, in a manner unsuspected till now. Besides this, we have obtained a connection between h and the Rogers dilogarithm function, through the fractal index defined in terms of the partition function associated with the universal class h of particles. Thus, considering the Eqs.(9, 10) and the Eq.(14), we have

$$\frac{L[x^\nu]}{L[1]} = i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right], \quad \nu = 0, 2, 4, \text{etc.} \quad (16)$$

$$\frac{L[x^\nu]}{L[1]} = 2 \times i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right], \quad \nu = 1, 3, 5, \text{etc.} \quad (17)$$

Also in [1] we have established a connection between the fractal parameter h and the Farey series of rational numbers, therefore once the classes h satisfy all the properties of these series we have an infinity collection of them. In this sense, we clearly establish a connection between number theory and the Rogers dilogarithm function. Given that the fractal parameter is an irreducible number $h = \frac{p}{q}$, the classes satisfy the properties [7]

P1. If $h_1 = \frac{p_1}{q_1}$ and $h_2 = \frac{p_2}{q_2}$ are two consecutive fractions $\frac{p_1}{q_1} > \frac{p_2}{q_2}$, then $|p_2 q_1 - q_2 p_1| = 1$.

P2. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ are three consecutive fractions $\frac{p_1}{q_1} > \frac{p_2}{q_2} > \frac{p_3}{q_3}$, then $\frac{p_2}{q_2} = \frac{p_1 + p_3}{q_1 + q_3}$.

P3. If $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are consecutive fractions in the same sequence, then among all fractions between the two, $\frac{p_1 + p_2}{q_1 + q_2}$ is the unique reduced fraction with the smallest denominator.

For example, consider the Farey series of order 6, denoted by the ν sequence

$$\begin{aligned} (h, \nu) = & \left(\frac{11}{6}, \frac{1}{6}\right) \rightarrow \left(\frac{9}{5}, \frac{1}{5}\right) \rightarrow \left(\frac{7}{4}, \frac{1}{4}\right) \rightarrow \left(\frac{5}{3}, \frac{1}{3}\right) \rightarrow \\ & \left(\frac{8}{5}, \frac{2}{5}\right) \rightarrow \left(\frac{3}{2}, \frac{1}{2}\right) \rightarrow \left(\frac{7}{5}, \frac{3}{5}\right) \rightarrow \left(\frac{4}{3}, \frac{2}{3}\right) \rightarrow \\ & \left(\frac{5}{4}, \frac{3}{4}\right) \rightarrow \left(\frac{6}{5}, \frac{4}{5}\right) \rightarrow \left(\frac{7}{6}, \frac{5}{6}\right) \rightarrow \dots \end{aligned} \quad (18)$$

Using the fractal spectrum (Eq.5), we can obtain other sequences which satisfy the Farey properties and for the classes

$$h = \frac{11}{6}, \frac{9}{5}, \frac{7}{4}, \frac{5}{3}, \frac{8}{5}, \frac{3}{2}, \frac{7}{5}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \dots,$$

and (note that these ones are dual classes, $\tilde{h} = 3 - h$) we can calculate the fractal index taking into account the Rogers dilogarithm function or the partition function associated with each h .

In summary, we have obtained a connection between fractons and CFT-quasiparticles. This was implemented with the notion of the *fractal index* associated with the universal class h of the fractons. This way, fractons and CFT-quasiparticles satisfy a specific fractal statistics. A connection between Rogers dilogarithm function, Farey series of rational numbers and Hausdorff dimension h , also was established. The idea of fractons as quasiparticles has been explored in the contexts of the fractional quantum Hall effect [1], high- T_c superconductivity [8] and Luttinger liquids [9]. A connection between fractal statistics and black hole entropy also was exploited in [10]. Finally, a fractal-deformed Heisenberg algebra for each class of fractons was introduced in [11].

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